Quasistatic calculation of linear response for crack propagation

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Quasistatic calculation using Mushkelishvili's analytic functions is presented for the linear-response function of the mode I (opening mode) crack shape to externally applied stress in order to examine the importance of the detailed structure of the cohesive zone that has been found in the fully dynamical calculation recently done by Ching, Langer, and Nakanishi [Phys. Rev. E **53**, 2864 (1996)]. It is confirmed that the result does not reduce to that of Cotterell and Rice [Int. J. Fract. **16**, 155 (1980)] if the cohesive shear stress is included. [S1063-651X(96)50911-6]

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The stability of crack propagation has attracted much attention since the recent experiments [1–4] and numerical simulations [5–8] that showed that straight crack propagation becomes unstable when the crack speed reaches a certain critical value. It has been noted by Yoffe that there should be such an instability because the angular dependence of the diverging part of stress around the crack tip has a maximum amplitude about 60° away from the direction in which the crack is moving when the crack speed exceeds about 60% of the Rayleigh speed. This, however, does not prove the existence of the instability because the analysis is not about the actual stress around the crack tip, but only about the diverging part of the stress, and the crack motion is not examined dynamically in the way that the crack path can be calculated.

The stability of straight crack propagation has been analyzed in the quasistatic limit by Cotterell and Rice (CR) [9]. They employed the continuum model without a cohesive zone; thus the stress diverges at the crack tip in the model. They determined the crack path using the condition that the shear component of the stress intensity factor should be zero and derived the criterion that straight crack propagation becomes unstable when the nondivergent component of tangential stress T around the crack tip becomes positive. The crack path $Y(x)$ was predicted as

$$Y(x)\sim\begin{cases}\exp\left[\dfrac{8T^2}{K^2}|x|\right] & (\text{for } T>0)\\ \sqrt{|x|} & (\text{for } T<0)\end{cases} \tag{1}$$

for $x<0$ when the crack propagating in the $-x$ direction deviates from the straight path at $x=0$; K denotes the stress intensity factor for mode I. Qualitatively similar results were reproduced by Adda-Bedia and Amar [10] recently.

On the other hand, Ching, Langer, and Nakanishi (CLN) [11] recently performed fully dynamical analysis on the continuum model, but with the cohesive zone, which is to suppress the unphysical divergence of the stress around the crack tip. For this system, stationary solutions of straight crack propagation had been obtained by Langer and Nakanishi [12] and Ching [13], and CLN calculated its linear response of the crack shape to the external shear stress. They showed that (i) the straight crack propagation is always unstable in the conventionally used continuum model, where the simple cohesive stress with only a normal component is assumed, and the crack tends to deviate strongly in the length scale of the cohesive zone for any crack speed $v>0$, and (ii) the shear component of the cohesive stress plays an important role in the stability.

Their results suggest that the stability depends on the detailed structure of the cohesive zone, which seems to conflict with the traditional idea that crack propagation can be analyzed by examining the field only outside the crack tip zone under the assumption that the microscopic details can be put into a few phenomenological parameters such as surface energy.

Their results also present a problem when one compares it with the calculation by CR [9] since CR ignored the size of the cohesive zone in their analysis, but CLN's result of the response function $\chi_Y(m,v)$ with the central cohesive stress has a singularity at the crack speed $v=0$ and the size of cohesive zone $l=0$:

$$\chi_Y^{-1}(m,v)\approx\frac{K}{2\sqrt{2}}(-im)^{3/2}\frac{(1-1/\kappa)v^2/2+iml/2}{(1-1/\kappa)v^2/2-iml}\quad \text{for } v,ml\ll 1, \tag{2}$$

where m and κ are the wave number of the perturbation and a material parameter given by the square of the ratio of the longitudinal wave speed to the transverse wave speed $\kappa=(c_l/c_t)^2$, respectively. (The extra factor $1/2\sqrt{2}$ comes from the definition of the stress intensity factor.) Thus, if one assumes the cohesive zone with finite size, the quasistatic limit is different from the case where one neglects the size of the cohesive zone from the beginning. It is very difficult, however, to compare those two calculations directly because CLN's calculation is fully dynamic and mathematically very complicated, and is very different from CR's.

The purpose of this report is to present a CR-type calculation for the model with the cohesive zone and to see if one can reproduce CLN's results by means of a completely different method. By doing this, one can see the relationship between the two calculations.

Using Mushkelishvili's analytic functions [15], the two-dimensional displacement field $\vec{u}\equiv(u,v)$ and the stress field $\Sigma_{ij}\equiv\sigma_{ij}/2\mu$ are expressed as

$$u(x,y)+iv(x,y)=\frac{\kappa+1}{\kappa-1}\phi(z)-z\,\overline{\phi'(z)}-\overline{\psi(z)},\qquad(3)$$

$$\Sigma_{xx}(x,y)=\mathrm{Re}[2\phi'(z)-\bar{z}\phi''(z)-\psi'(z)],\qquad(4)$$

$$\Sigma_{yy}(x,y)=\mathrm{Re}[2\phi'(z)+\bar{z}\phi''(z)+\psi'(z)],\qquad(5)$$

$$\Sigma_{xy}(x,y)=\mathrm{Im}[\bar{z}\phi''(z)+\psi'(z)],\qquad(6)$$

where $\phi(z)$ and $\psi(z)$ are analytic functions with $z=x+iy$ and μ is a shear modulus. To study the crack propagation, we consider the situation where the material is being pulled by the stress tensor

$$\Sigma_\infty\equiv\begin{pmatrix}\Sigma_{T\infty} & 0\\ 0 & \Sigma_{N\infty}\end{pmatrix}\qquad(7)$$

at infinity. For a technical reason, in the following calculation we consider the situation where $-\Sigma_\infty$ is acting on the crack surface to cancel the stress (7) at infinity, which is equivalent to the above situation within the linear elasticity.

In order to avoid unphysical divergence of stress at the crack tip, we introduce the simple cohesive zone with the constant stress Σ_0, the range of the stress δ, and the linear size l, which gives the surface energy $\gamma=\delta\Sigma_0$. We will see that *Griffith's condition* [14] *for cracking and the response function for the crack shape can be derived from the condition that the stress should not diverge at the crack tip.*

(i) Zeroth-order calculation. Let us first consider the case where the straight crack is along the positive part of the real axis without the external shear perturbation (the zeroth order). Then the boundary condition for the normal stress $\Sigma^b_{N0}(x)$ and the shear stress $\Sigma^b_{S0}(x)$ along the crack surface is

$$\Sigma^b_{N0}(x)=-\Sigma_{N\infty}\theta(L-x)+\Sigma_0\theta(l-x),\quad \Sigma^b_{S0}(x)=0,\qquad(8)$$

where $\theta(x)$ is the step function and the length scale L is introduced to avoid the infinite stress intensity factor. L roughly corresponds to the width of the stripe in a real situation, but it is more convenient to replace it with the stress intensity factor K, as we will do in the following. Then, the zeroth-order analytic function $\phi_0(z)$ is given by

$$\begin{aligned}\phi_0'(z)&=\frac{1}{2\pi\sqrt{-z}}\int_0^\infty dt\,\frac{\sqrt{t}}{z-t}\Sigma^b_{N0}(t)\\&=-\frac{1}{2\pi}\Bigg\{\Sigma_{N\infty}\Bigg[-i\Bigg(\ln\frac{\sqrt{L}-i\sqrt{-z}}{\sqrt{L}+i\sqrt{-z}}+i\pi\Bigg)-\frac{2\sqrt{L}}{\sqrt{-z}}\Bigg]\\&\quad-\Sigma_0\Bigg[-i\Bigg(\ln\frac{\sqrt{l}-i\sqrt{-z}}{\sqrt{l}+i\sqrt{-z}}+i\pi\Bigg)-\frac{2\sqrt{l}}{\sqrt{-z}}\Bigg]\Bigg\}.\end{aligned}\qquad(9)$$

The condition of no stress divergence tells us that the coefficient of $1/\sqrt{-z}$ should be zero:

$$\sqrt{L}\Sigma_{N\infty}=\sqrt{l}\,\Sigma_0.\qquad(10)$$

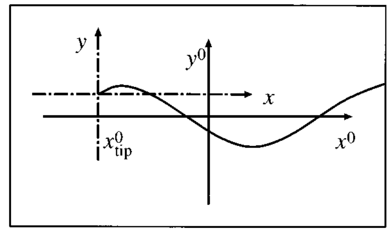

FIG. 1. The x^0-y^0 coordinate fixed to the material and the x-y coordinate moving with the crack tip.

Thus we have for $\phi_0(z)$ and $\psi_0(z)$

$$\begin{aligned}\phi_0'(z)=\frac{-1}{2\pi}\Bigg\{&\pi(\Sigma_{N\infty}-\Sigma_0)-i\Sigma_{N\infty}\ln\Bigg(\frac{\sqrt{L}-i\sqrt{-z}}{\sqrt{L}+i\sqrt{-z}}\Bigg)\\&+i\Sigma_0\ln\Bigg(\frac{\sqrt{l}-i\sqrt{-z}}{\sqrt{l}+i\sqrt{-z}}\Bigg)\Bigg\},\end{aligned}\qquad(11)$$

$$\psi_0'(z)=-z\phi_0''(z).\qquad(12)$$

By examining the stress field around the crack tip in the region $l\ll|z|\ll L$, the stress intensity factor K in the present situation is obtained as

$$K=2\sqrt{\frac{2}{\pi}}\sqrt{L}\;\Sigma_{N\infty}\qquad(13)$$

and the no-stress-divergence condition (10) is expressed as

$$\sqrt{\frac{\pi}{2}}K=2\sqrt{l}\;\Sigma_0.\qquad(14)$$

The half of the crack opening $U_N(x)$ defined as

$$U_N(x)\equiv u(x,y=+0)\qquad(15)$$

should satisfy the matching condition $U_N(l)=\delta$. This relation allows us to express the no-divergence condition (14) in the form of Irwin's criterion [16]

$$K^2=4\frac{\kappa-1}{\kappa}\gamma,\qquad(16)$$

which is equivalent to Griffith's condition [14]. In the small-x limit, the zeroth-order crack opening $U_{N0}(x)$ is given by

$$U_{N0}(x)\approx\frac{2}{3}\delta\left(\frac{x}{l}\right)^{3/2}\equiv\overline{U_{N0}}\left(\frac{x}{l}\right)^{3/2}\quad\text{for }x\ll l,\qquad(17)$$

which we will use later.

(ii) First-order calculation. Now we apply perturbation $\varepsilon_{ex}(x)$ in the shear stress on the crack surface and examine the response of the crack path $y=Y(x)$ within the linear approximation. We introduce two coordinate systems: the x^0-y^0 system that is fixed to the material and the x-y system that is moving with the crack and its origin is located at the crack tip (Fig. 1), and we assume

$$\varepsilon_{ex}(x)=\mathrm{Re}[\hat{\varepsilon}_m e^{im(x+x^0_{\mathrm{tip}})}],$$

$$Y(x)=\mathrm{Re}[\hat{Y}_m(e^{imx}-1)e^{imx^0_{\mathrm{tip}}}], \tag{18}$$

where x^0_{tip} is the x^0 coordinate of the crack tip in the x^0-y^0 system. The normal and tangential vectors to the crack surface are given by

$$\vec{n}(x)\equiv\frac{(-Y'(x),1)}{\sqrt{1+Y'(x)^2}},\quad \vec{t}(x)\equiv\frac{(1,Y'(x))}{\sqrt{1+Y'(x)^2}}. \tag{19}$$

Thus the boundary conditions for the normal and the shear stress along the crack surface up to the linear in Y are

$$\begin{aligned}\Sigma^b_{N1}(x)&=-(\vec{n}\Sigma^\infty\vec{n})\ \theta(L-x)+\Sigma_0\ \theta(l-x)\\&=-\Sigma_{N\infty}\ \theta(L-x)+\Sigma_0\ \theta(l-x)=\Sigma^b_{N0}(x),\end{aligned} \tag{20}$$

$$\begin{aligned}\Sigma^b_{S1}(x)&=-(\vec{n}\Sigma^\infty\vec{t})\ \theta(L-x)+\varepsilon_{ex}(x)+\Sigma_{cS}\ \theta(l-x)\\&=-\Delta\Sigma_\infty Y'(x)\ \theta(L-x)+\varepsilon_{ex}(x)+\Sigma_{cS}\ \theta(l-x),\end{aligned} \tag{21}$$

where $\Delta\Sigma_\infty\equiv\Sigma_{N\infty}-\Sigma_{T\infty}$ and Σ_{cS} is the shear component of the cohesive stress, which we will discuss later.

In order to obtain the boundary condition for the first-order functions $\phi_1(z)$ and $\psi_1(z)$, we need to subtract from Eqs. (20) and (21) the stress that is induced by the zeroth-order solution along the crack surface at $y=Y(x)$:

$$\Sigma^0_{nn}(x,Y(x))\approx\Sigma^b_{N1}(x),$$

$$\Sigma^0_{tn}(x,Y(x))\approx-2\mathrm{Re}[\phi_0''(x,+0)]\ Y(x). \tag{22}$$

Thus

$$\begin{aligned}\phi_1'(z)&=\frac{-i}{2\pi\sqrt{-z}}\int_0^\infty dt\frac{\sqrt{t}}{z-t}[\Sigma^b_{S1}(t)-\Sigma^0_{tn}(t)]\\&=\frac{-i}{2\pi\sqrt{-z}}\bigg[-\frac{\sqrt{l}}{z-l}\Sigma_0 Y(l)-\int_0^\infty dt\frac{\sqrt{t}}{z-t}\\&\quad\times[\Delta\Sigma_\infty Y'(t)-\varepsilon_{ex}(t)]\\&\quad+\int_0^l dt\frac{\sqrt{t}}{z-t}\Sigma_{cS}(t)\bigg],\end{aligned} \tag{23}$$

$$\psi_1'(z)=-z\phi_1''(z)-2\phi_1'(z), \tag{24}$$

where we have taken the $L\to\infty$ limit when the integral converges.

(a) The case without shear stress. First, we consider the case without shear cohesion: $\Sigma_{cS}=0$. Then, the condition of no stress divergence in (23) gives

$$\frac{\hat{E}_m}{\hat{Y}_m}-(\sqrt{-im})^3\sqrt{L}\frac{\Sigma_{N\infty}}{\sqrt{\pi}}\frac{e^{iml}-1}{iml}=0, \tag{25}$$

where $\hat{E}_m\equiv im\Delta\Sigma_\infty\hat{Y}_m+\hat{\varepsilon}_m$. Thus the response function $\chi_Y(m)$ is given by

$$\chi_Y^{-1}(m)\equiv-\frac{\hat{\varepsilon}_m}{\hat{Y}_m}=-im\Delta\Sigma_\infty+(\sqrt{-im})^3\left(\frac{K}{2\sqrt{2}}\right)\frac{e^{iml}-1}{iml}, \tag{26}$$

where we have inserted the minus sign in the definition of $\chi_Y(m)$ in order to make a connection with CLN, which comes from the difference in the way the perturbation is imposed. In the work of CLN, the perturbation is ahead of the crack, while it is on the crack surface in the present calculation. It can be shown that (5.21) of CLN without the $\widetilde{\mathcal{D}}_1$ term reduces to (26) in the $v=0$ limit. The expression in the $ml\to0$ limit,

$$\chi_Y^{-1}(m)\approx-im\Delta\Sigma_\infty+(\sqrt{-im})^3\left(\frac{K}{2\sqrt{2}}\right), \tag{27}$$

gives CR's result if one notes that $-\Delta\Sigma_\infty$ corresponds to T stress of CR's paper [9] in the present configuration.

(b) The case with shear stress. Now we suppose the shear part of the cohesive force is nonzero and given by

$$\Sigma_{cS}(x)=\Sigma_c(|\vec{U}(x)|)\frac{U_S(x)}{|\vec{U}(x)|}(1+\rho);$$

$$\Sigma_{cN}(x)=\Sigma_c(|\vec{U}(x)|)\frac{U_N(x)}{|\vec{U}(x)|}, \tag{28}$$

where $U_S(x)$ and $U_N(x)$ are the shear and the normal part of the crack opening. In the case of $\rho=0$, the cohesive stress is ''central force.''

Then we have, for the nondivergence condition in (23),

$$\frac{\Sigma_0}{\sqrt{l}}Y(l)+\int_0^\infty dt\frac{1}{\sqrt{t}}[\Delta\Sigma_\infty Y'(t)-\varepsilon_{ex}(t)]-\int_0^l\frac{1}{\sqrt{t}}\Sigma_{cS}(t)=0 \tag{29}$$

and $\phi_1(z)$ becomes

$$\begin{aligned}\phi_1'(z)=\frac{i\sqrt{-z}}{2\pi}\bigg[&-\frac{\Sigma_0 Y(l)}{(z-\ell)\sqrt{l}}-\int_0^\infty dt\frac{\Delta\Sigma_\infty Y'(t)-\varepsilon_{ex}(t)}{(z-t)\sqrt{t}}\\&+\int_0^l dt\frac{\Sigma_{cS}(t)}{(z-t)\sqrt{t}}\bigg].\end{aligned} \tag{30}$$

In the first order of $\hat{\varepsilon}_m$, $\Sigma_{cS}(x)$ should have the form

$$\Sigma_{cS}(x)=\hat{\sigma}_{cS1m}e^{im(x+x^0_{\mathrm{tip}})}. \tag{31}$$

Then (29) gives

$$\begin{aligned}&\frac{\hat{E}_m}{\hat{Y}_m}-(\sqrt{-im})^3\left(\frac{K}{2\sqrt{2}}\right)\frac{e^{iml}-1}{iml}\\&\quad-\frac{2\sqrt{ml}}{\sqrt{i}\sqrt{\pi}}\frac{1}{\sqrt{ml}}\int_0^{\sqrt{ml}}dt\ e^{it^2}\left(\frac{\hat{\sigma}_{cS1m}}{\hat{Y}_m}\right)=0.\end{aligned} \tag{32}$$

$\hat{\sigma}_{cS1m}$ can be eliminated by using (28), where the shear part of the crack opening $U_S(x)$ defined by

$$U_S(x) \equiv \frac{1}{2}[\vec{u}(x,Y(x)+0) - \vec{u}(x,Y(x)-0)] \cdot \vec{t} \qquad (33)$$

can be estimated up to the first order for $x \ll l$ as

$$U'_{S1}(x) = \frac{1}{2\pi}\frac{2\kappa}{\kappa-1}\frac{1}{l}\sqrt{\frac{x}{l}}\{[(e^{iml}-1)(1-2iml)-iml] \times \Sigma_0 \hat{Y}_m + 2le^{iml}\hat{\sigma}_{cS1m}\}e^{imx^0_{\rm tip}}, \qquad (34)$$

using (30) and (11). From (32), (28), (17), and (34), we obtain the expression

$$\chi_Y^{-1}(m) = -im\Delta\Sigma_\infty + (\sqrt{-im})^3\left(\frac{K}{2\sqrt{2}}\right)\left(\frac{e^{iml}-1}{iml} + \frac{2e^{iml}-(e^{iml}-1)/iml-1}{1-e^{iml}(1+\rho)}\frac{1}{\sqrt{ml}} \times \int_0^{\sqrt{ml}} dt\, e^{it^2}(1+\rho)\right)$$
$$\approx (\sqrt{-im})^3\left(\frac{K}{2\sqrt{2}}\right)\frac{iml/2-\rho}{-iml-\rho} \quad \text{for } ml,\rho \ll 1, \qquad (35)$$

where we have ignored the $\Delta\Sigma_\infty$ term in the last expression. This agrees exactly with (5.11) of CLN when (7.8) and (7.11) are estimated in the $v \to 0$ limit but all in order of ml.

There are several remarks on the results (26), (27), and (35).

(i) The stability of straight crack propagation can be determined by the pole of $\chi_Y(m)$ in the complex m plane. If there is a pole in the upper half plane, the system is unstable, if $\chi_Y(m)$ is analytic in the upper half plane, it is stable.

(ii) In these expressions of $\chi_Y^{-1}(m)$, the first term that contains $\Delta\Sigma_\infty$ can be ignored in comparison with the second term with the factor K when we consider the instability of the wave number $m \gg 1/L$ because $K \sim \sqrt{L}\Sigma_{N\infty}$.

(iii) The present calculation is quasistatic and corresponds to the $v=0$ limit of the dynamical result (2). Thus, in the $\rho=0$ case, the $ml=0$ limit of (35) gives the result obtained from (2) by taking the $m\ell=0$ limit *after* taking the $v=0$ limit. It should be noted that this gives a result different from (27) for the case without shear cohesion, or CR's case.

(iv) The results with and without shear cohesion have very different physical contents for general ρ: From (27) of the no-shear-cohesion case, the stability appears to be controlled by $\Delta\Sigma_\infty$, or T stress, while the result (35) with shear cohesion shows that the straight crack propagation is strongly unstable with the length scale l for $\rho<0$, namely, for the case where the shear cohesion is weaker than the normal cohesion. This is a contradiction because the no-shear-cohesion case should be included in the case $\rho<0$. It can be shown numerically, however, that the dynamical result $\chi_Y(m,v)$ in the no-shear-cohesion case, namely, (5.21) in the work of CLN without the $\widetilde{\mathcal{D}}_1$ term, has an unstable pole that goes to infinity as $v \to 0$. This suggests that the $v=0$ limit is singular in the case without shear cohesion and the quasistatic calculation cannot determine the stability correctly.

(v) On the other hand, in the $\rho>0$ case, Eq. (35) shows stability in the microscopic length scale. Thus, in certain situations the expression (35) in the small ml limit is an appropriate one and it would predict the same trajectory with (27) in the macroscopic level.

In summary, the CR-type calculation ignoring the cohesive zone can only determine the macroscopic stability provided the system is microscopically stable and the microscopic stability depends on the detailed structure of the cohesive zone. The ordinary model with only the normal cohesive stress is microscopically unstable and one way to achieve the microscopic stability is stronger cohesive shear stress.

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